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Abstract:

This deliverable contains a publication (currently rejected and to be submitted) on the analytical solution of the weight change of the time-continuous version of spike-timing-dependent plasticity which is a biologically realistic learning rule. For the first time it is now possible to calculate plasticity analytically for continuously changing inputs. This is of relevance for all behaving systems (machines, robots, animals) which interact with their environment leading to widely varying neural activation.

Keyword list: Spike-timing-dependent plasticity, non-stationary inputs, multi-input systems.

Scientific Publication on the analytical solution for the weight growth in time-continuous spike-timing-dependent plasticity.

Wörgötter F. and Kolodziejski, C.

Executive Summary

This deliverable consists of a paper (Kolodziejski and Wörgötter, 2009) which derives an analytical solution to multi-synapse systems under Hebbian plasticity rules which are the time-continuous correlate of spike-timing-dependent plasticity. This paper had originally been submitted to Phys. Rev. Letters, but recently been rejected. It will be resubmitted to a different journal in due time (before the end of PACO+).

Background: The theory of neural networks has become a large and influential field not only in neural computation but was also applied to many different areas like for instance to machine learning or to robotics. It is, thus, intriguing that several, quite fundamental aspects remain unresolved, in particular, the temporal dynamics of these systems. For example, so far it has not been possible for most networks to analytically calculate the temporal development of synaptic weights from known input patterns particularly if those patterns are temporally changing in a complex way. This generically applies to all systems (animals, machines, robots, etc.) which interact with their environment as their own behavior will lead to continuously changing inputs and, thus, to an ongoing synaptic weight change. So far the investigation of such non-stationary systems relied fundamentally on numerical calculations or estimations.

Here we have derived an analytical solution for the biological realistic class of linear, correlation based ("Hebbian") learning rules. In these rules the change of a synapse follows the correlation between pre- and postsynaptic activity at a neuron. Using the derivative of the postsynaptic input leads directly to the spike-timing-dependent plasticity, however, with the advantage of time-continuous input patterns. Another very important feature of correlation based rules is the straightforward usage of many inputs at the same time which is a compulsory requirement for an adaptive system that deals with closed-loop systems.

Novel Contribution: This is to our knowledge the first time that the plasticity of systems described above can be calculated analytically. This makes it possible to overcome the need of numerical calculations and expansive robotic simulations.

One possibility in which neural networks control closed-loop systems is by their dynamics. However, usually the synaptic wiring of such networks is fixed and the tuning of those connections is a demanding task. With the analytical method developed in this paper we will next investigate arbitrary networks in their dynamical behavior which allows us to define the parameters in an appropriate way leading to specific robot behavior.

Appendix

Christoph Kolodziejski and Florentin Wörgötter (2009) Plasticity of many-synapse systems. (originally submitted to PRL, the rejected and planned for a submission at some other journal soon).

Plasticity of many-synapse systems

C. Kolodziejski* and F. Wörgötter

*Bernstein Center for Computational Neuroscience, Göttingen
Georg-August-University Göttingen, Germany*

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The dynamics of neural systems is influenced by synaptic plasticity which alters the network connections. Better analytical understanding of plasticity is needed for predicting learning and adaptation in such systems. Here we derive a non-stationary solution of the weight development of many synapses for correlation based learning rules. For the first time it is now possible to calculate plasticity analytically for continuously changing inputs. This is of relevance for all behaving systems (machines, animals) which interact with their environment leading to widely varying neural activation.

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The theory of neural networks, developed since around 1943 [1], has become a large and influential field in physics as well as other disciplines. It is, thus, intriguing that several, quite fundamental aspects remain unresolved, in particular, the temporal dynamics of these systems. For example, so far it has not been possible for most networks to analytically calculate the temporal development of synaptic weights from known input patterns. Specific solutions for certain types of networks and learning rules have been provided, all of which however need to constrain structure or dynamics of the system in different ways [2–8]. Constraining the dynamics often appears to be problematic as most neural networks receive complex temporally changing input patterns, which lead to an ongoing weight change. So far such non-stationary systems can only be treated numerically.

Here we will focus on the large class of linear, correlation based (“Hebbian”) learning rules. In these rules the change of a synapse follows the correlation between pre- and postsynaptic activity [9] at a neuron. They are probably the most basic class of all network learning rules and related to plasticity in the brain [10]. The aim of this paper is to present an analytical, non-stationary solution of Hebbian plasticity for arbitrary numbers of changing synapses at one given neuron.

With this method ongoing synaptic plasticity can be calculated for temporally changing inputs. This is of high relevance as it is known that in behaving animals sensory inputs are highly non-stationary [11]. This generically applies to all systems (animals, machines, robots, etc.) which interact with their environment as their own behavior will lead to continuously changing inputs and, thus, to an ongoing synaptic weight change. This comprises a very large and important class of dynamic systems and the solution provided here may allow for the first time calculating Hebbian plasticity in such systems without restrictions.

The general system is shown in Fig. 1 on the right side. It consists of N synapses with strength ω_i that receive input from neurons i with its continuous value x_i .

Each input produces an excitatory post synaptic potential (EPSP) which is modeled by Kernel functions h_i (see inset in Fig. 2 for an example). The output of the neuron is, thus:

$$v(t) = \sum_{i=1}^N (x_i * h_i)(t) \cdot \omega_i(t) \quad (1)$$

where $(\xi * \eta)(t) = \int_0^\infty \xi(\tau) \eta(t - \tau) d\tau$ describes a convolution.

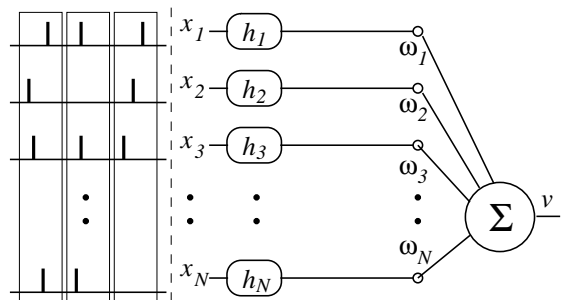


FIG. 1: This figure shows our general setup and example input values that are fed to the neuron. Inputs are denoted as x_i , Kernel functions as h_i , synaptic strength as ω_i and the output of the model neuron as v . The example inputs shown on the left side are spike trains, although any arbitrary continuous function can serve as an input.

The synapses change according to a general formalized Hebbian plasticity rule

$$\dot{\omega}_i(t) := \frac{d\omega_i(t)}{dt} = \mu F[x_i * h_i](t) G[v](t) \quad (2)$$

where μ is the plasticity rate and $F[\cdot]$ and $G[\cdot]$ are linear functionals.

Important are conventional Hebbian learning with $F = G = \mathbf{1}$ (where $\mathbf{1}$ is the identity) and differential Hebbian learning [12], which allows for the learning of temporal sequences of input events [13], with $F = \mathbf{1}$ and $G = \frac{d}{dt}$.

To avoid that weight changes will follow spurious random correlations one generally assumes that learning is a slow process, where inputs change much faster than weights, with $\frac{\omega'_i}{\omega_i} \ll \frac{(x_i * h_i)'}{x_i * h_i}$, $\mu \rightarrow 0$. This simplifies Eq. 2 and we neglect all temporal derivatives of ω_i on the right hand side:

$$\dot{\omega}_i(t) = \mu F[x_i * h_i](t) \sum_{j=1}^N \omega_j(t) G[x_j * h_i](t) \quad (3)$$

where we used $G[\sum \xi_i] = \sum G[\xi_i]$ as $G[\cdot]$ is linear.

If we take ω_i as the i -th component of a vector $\boldsymbol{\omega}$, we write

$$\dot{\boldsymbol{\omega}}(t) = \mu \mathbf{A}(t) \boldsymbol{\omega}(t) \quad (4)$$

with $A_{ij}(t) = F[x_i * h_i](t) G[x_j * h_i](t)$ or in matrix form $\mathbf{A}(t) = F[\mathbf{x} * \mathbf{h}](t) \cdot G[\mathbf{x} * \mathbf{h}](t)$ where $\bar{\boldsymbol{\xi}}$ denotes the transposition of matrix $\boldsymbol{\xi}$.

The solution of Eq. 4 is not trivial as the matrix $\mathbf{A}(t)$ is also a function of time. This problem is often found in quantum mechanics and the main problem is that matrices usually do not commute. However, there exist a solution which includes an infinite series, called the Magnus series (see [14] for more details), with

$$\boldsymbol{\omega}(t) = \exp \boldsymbol{\Omega}(t) \cdot \boldsymbol{\omega}_0 \quad (5)$$

where $\boldsymbol{\omega}_0$ is the synaptic strength before plasticity and $\boldsymbol{\Omega}(t)$ is the solution of following equation

$$\dot{\boldsymbol{\Omega}}(t) = \left\{ \mu \mathbf{A}(t), \frac{\boldsymbol{\Omega}(t)}{1 - \exp(-\boldsymbol{\Omega}(t))} \right\} = \sum_{n=0}^{\infty} \beta_n \{ \mathbf{A}, \boldsymbol{\Omega}^n \}. \quad (6)$$

Here the braces $\{\eta, \xi^n\} = [\dots[[\eta, \xi], \xi] \dots \xi]$ are nested commutators $[\eta, \xi] = \eta \xi - \xi \eta$ and β_n are the coefficients of the Taylor expansion of $\frac{\boldsymbol{\Omega}}{1 - \exp(-\boldsymbol{\Omega})}$ around $\boldsymbol{\Omega} = 0$. Eq. 6 is solved through integration by iteration to the Magnus series:

$$\begin{aligned} \boldsymbol{\Omega}(t) &= \mu \boldsymbol{\alpha}(t) + \frac{\mu^2}{2} \int_0^t [\mathbf{A}(\tau), \boldsymbol{\alpha}(\tau)] d\tau \\ &+ \frac{\mu^3}{4} \int_0^t \left[\mathbf{A}(\tau), \int_0^\tau [\mathbf{A}(\sigma), \boldsymbol{\alpha}(\sigma)] d\sigma \right] d\tau \\ &+ \frac{\mu^3}{12} \int_0^t [[\mathbf{A}(t), \boldsymbol{\alpha}(\tau)], \boldsymbol{\alpha}(\tau)] d\tau \\ &+ o(\mu^4) \end{aligned} \quad (7)$$

with $\boldsymbol{\alpha}(t) = \int_0^t \mathbf{A}(\tau) d\tau$. Thus, Eq. 5 combined with Eq. 7, gives us analytically the time development of all weights connected to a neuron under Hebbian plasticity in the limit of small plasticity rates μ . With this we are able to calculate without simulations in principle directly the synaptic strengths of N synapses given N different spike trains, membrane potentials, or firing rates.

Next we transform the solution into a computable form and provide error estimates. As the commutators in the infinite series in Eq. 7 are generally non-zero we truncate the series and neglect iterations above degree (k). We write the truncated solution as:

$$\boldsymbol{\omega}_{(k)}(t) = \exp \boldsymbol{\Omega}_{(k)}(t) \cdot \boldsymbol{\omega}_0 \quad (8)$$

For two synapses this is solved directly in the appendix, most often, however this needs to be calculated by expanding the exponential function. We denote this approximation with a prime, i.e. (k')

$$\boldsymbol{\omega}_{(k')}(t) = \left(\mathbf{I} + \sum_{p=2, q=1}^{p \cdot q \leq k} (\boldsymbol{\Omega}_{(p)}(t))^q \right) \cdot \boldsymbol{\omega}_0 = \boldsymbol{\mathfrak{B}}_{(k')}(t) \cdot \boldsymbol{\omega}_0 \quad (9)$$

where \mathbf{I} is the identity matrix and $\boldsymbol{\mathfrak{B}}_{(k')}(t)$ the transformation of order (k) from the initial synaptic strength $\boldsymbol{\omega}_0$ to the synaptic strength at time t . Notice that in the limit $k \rightarrow \infty$ the approximation (Eq. 8) transforms into the general solution (Eq. 5). This solution is computable for arbitrary input patterns.

Now as we know the complete analytical solution of Eq. 4 we investigate approximations and their errors in order to judge their usefulness for further considerations. Therefore, we will use in the following spikes as the inputs to the system and assume that all $h_i = h$ are equal. The spikes are modeled as delta functions $\delta(t - t_i)$ for spike time t_i which simplifies the convolution to a temporal shift in the Kernel function h : $h(t - t_i) = \int_0^\infty \delta(t - t_i - \tau) h(\tau) d\tau$. This leads for elements of $\mathbf{A}(t)$ to $A_{ij}(t) = F[h](t - t_i) G[h](t - t_j)$ where t_i and t_j are the spike timings of neuron x_i and x_j respectively. We will use the Kernel, shown in the inset of Fig. 2, given by

$$h(t) = \frac{1}{\sigma} (e^{-\alpha t} - e^{-\beta t}) \Theta(t) \quad (10)$$

The different approximation errors are exemplified in Fig. 2. For this we are using a single spike pair at two synapses for which we calculate the final synaptic strength $\tilde{\boldsymbol{\omega}} = \lim_{\tau \rightarrow \infty} \boldsymbol{\omega}(\tau)$ (Eq. 5). This has been performed for differential Hebbian learning, but we remark that the error is identical for Hebbian learning. This is explained in the appendix where all details for the error calculations are provided. For this setup, weight changes are computed in three ways: without any approximations, yielding $\tilde{\boldsymbol{\omega}}$ (Eq. 5 and Eq. 7); using the truncated solution only, yielding $\tilde{\boldsymbol{\omega}}_{(k)}$ (Eq. 8); and using the truncated solution while also expanding the exponential function, yielding $\tilde{\boldsymbol{\omega}}_{(k')}$ (Eq. 9). Thus, we use $\tilde{\boldsymbol{\omega}}$ and compare it to approximations $\tilde{\boldsymbol{\omega}}_{(\cdot)}$, calculating the error as: $\Delta_{(\cdot)} = |\tilde{\boldsymbol{\omega}}_{(\cdot)} - \tilde{\boldsymbol{\omega}}|$. This is plotted in Fig. 2 for different approximations against the plasticity rate μ on a log-log scale where we set the timescale of the input Kernel h to 1. As approximations (k) and (k') become very similar for $k > 2$ only four curves are shown. We observe

that the behavior of the difference-error $\Delta_{(\cdot)}$ follows the order of the approximation used. The error for the linear expansion approximation ($k = 2'$, Eq. 9) is slightly higher than that from its corresponding truncation approximation ($k = 2$, Eq. 8). However, using a plasticity rate of $\mu = 0.001$ which already results in a difference-error value of 10^{-8} compared to 10^{-2} when using the same timescale for μ and h . Therefore one can in most applications use even the simplest possible linear approximation ($k = 2'$) to calculate the change in synaptic strength.

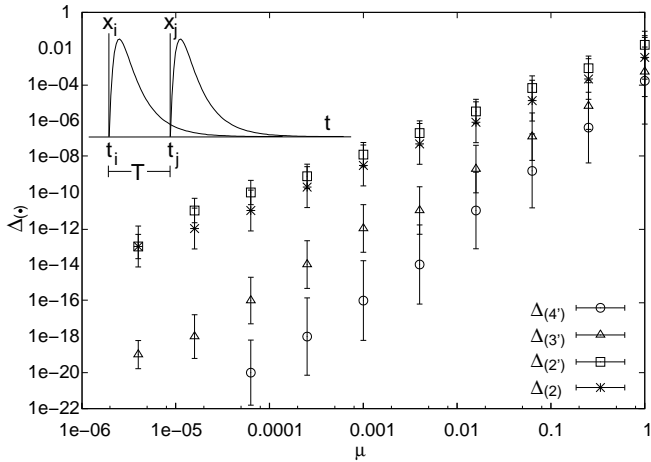


FIG. 2: Here we show the degree of consistency between our general solution and the proposed approximations. To this end we plot the difference $\Delta_{(\cdot)}$ between the approximation and the exact solution of Eq. 4 for one input spike pair against the plasticity rate μ on a log-log scale. A kernel function h with $\alpha = 0.1$, $\beta = 0.2$, $\sigma = 0.25$ and $\max_t h(t) = 1$ is used. It is shown in the upper left of this figure for two spikes at times t_i and t_j . The temporal difference $T = t_2 - t_1$, between the two input spikes was varied over the length of the used Kernel functions (here between 1 and 100 steps) and error bars representing the standard deviation are given.

As this calculation has been based on two spikes at two synapses only, we need to ask how the error develops when using N synapses and complex spike trains. For this we first consider spike trains (see Fig. 1 left), which are grouped 'vertically' into groups with each input firing at most once. Kernels of spikes within a group will overlap but we assume that grouping is possible such that adjacent groups are spaced with a temporal distance sufficient to prevent overlap between kernel responses of temporally adjacent groups. Thus we calculate ω in the same way as above leading to: $\tilde{\mathfrak{B}}^{(k')} = \lim_{\tau \rightarrow \infty} \mathfrak{B}^{(k')}(\tau)$ in Eq. 9. Thus, when using such a temporal tiling, $\tilde{\mathfrak{B}}^{(k')}$ depends only on the spike timing matrix \mathbf{T} with elements $T_{ij} = t_j - t_i$. Then, we get the synaptic strength after M groups by calculating the product over all groups m :

$$\omega_{M,(k')} = \prod_{m=1}^M \tilde{\mathfrak{B}}^{(k')}(\mathbf{T}_m) \cdot \omega_0. \quad (11)$$

Physiologically such a grouping decomposition is performed for so-called non-bursting neurons, which, for example, constitute the majority of cortical cells. The solution (Eq. 11) is easy to compute. Because a product of matrices results in a summation of matrix elements, the error does not increase exponentially but only linearly in M . Because of this it follows that even after 10000 spikes the error is still of an order of only 10^{-4} given the example above (see Fig. 2).

Finally we estimate how the error behaves when kernels overlap. This mainly happens during bursts of spikes with temporarily high spiking frequencies, which are, in general, rare events. However, using the solution which assumes independent temporal intervals (Eq. 11) instead of the time-continuous calculation (Eq. 9) only includes an additional error of order ($k = 2$) due to the linearity of the Kernel functions h . The error after matrix multiplication (Eq. 11) results in the square of the lowest term of the Magnus series (Eq. 7).

Thus, the easily computable group decomposition suggested by Eq. 11 will yield accurate enough results even for long, non-bursting spike trains.

Real neurons often display rich, non-stationary, firing patterns by which all synaptic weights will be affected. The same is true for neurons in artificial neural networks, especially when being embedded in closed-loop (acting, behaving) systems. The so far existing solutions which describe Hebbian learning, on the other hand, constrain the temporal dynamics of the system or limit plasticity to a subset of synapses. With the solution presented here we calculate weight changes for the first time without these restrictions. This is a valuable step forward in our understanding of synaptic dynamics in different networks. Specifically, we have presented the time-continuous solution for the synaptic change of general Hebbian plasticity (Eq. 5 and Eq. 7), its approximation for general spiking or continuous inputs (Eq. 8 and Eq. 9) as well as a specific solution for non-bursting spike trains (Eq. 11). Of practical importance is the fact that the error of the computable approximations (Eqs. 8, 9, 11) remains small even for long spike trains.

The temporal development of multi-synapse systems and the conditions of stability are still not well understood. Some convergence conditions have been found (see for example [2-8]), however in general the synaptic strengths of such networks will diverge or oscillate. This is undesired, because network stability is important for the formation of (e.g.) stable memories or receptive fields. Using the time-continuous solution for linear Hebbian plasticity described here could serve as a starting point to better understand mechanisms, structures and conditions for which stable network configurations will emerge. The rich dynamics, which govern many closed-loop adaptive (network based) physical systems can, thus, now be better understood and predicted, which might have substantial future influence for the guided de-

sign of network controlled systems.

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Appendix: Here we calculate the solution for the two synapse system using differential Hebbian learning which is an important analytically fully solvable special case on which we have based the error analysis provide in the main text. Note, the order of the error is not affected by the actual (linear) Hebbian rule used as the only source of the error is the plasticity rate μ , which is independent of the choices of F and G .

If we now concentrate on $F = 1$ and $G = \frac{d}{dt}$ the matrix $\mathfrak{B}(t)$ results in

$$\mathfrak{B}(t) = \begin{pmatrix} 1 + \frac{1}{2}h^2(t) & \nu_T^{-1}(t) \\ \nu_T^{+1}(t) & 1 + \frac{1}{2}h^2(t-T) \end{pmatrix} \quad (12)$$

where $\nu_T^{-1}(t) = \int_0^t h(\tau) \dot{h}(\tau-T) d\tau$ and $\nu_T^{+1}(t) = \int_0^t h(\tau-T) \dot{h}(\tau) d\tau$.

Using the Kernel function h (Eq. 10) we analytically integrate the secondary diagonal entries of Eq. 12 which are:

$$\begin{aligned} \nu_T^{\kappa}(t) &= \frac{\Theta(t-T)\Theta(t)}{2(\alpha+\beta)\sigma^2} (\kappa \operatorname{sign}(T) \sigma (\alpha-\beta) h(|T|) \\ &\quad - 2e^{-t(\alpha+\beta)} (\alpha e^{\alpha T} + \beta e^{\beta T}) \\ &\quad + (\alpha+\beta)(e^{-\alpha(2t-T)} + e^{-\beta(2t-T)})). \end{aligned} \quad (13)$$

In the limit of t to infinity matrix $\mathfrak{B}(t)$ changes into $\tilde{\mathfrak{B}}$ and so do the secondary diagonal elements

$$\tilde{\nu}_T^{\kappa} = \lim_{t \rightarrow \infty} \nu_T^{\kappa}(t) = \kappa \operatorname{sign}(T) \frac{\alpha-\beta}{2(\alpha+\beta)\sigma} h(|T|) \quad (14)$$

and find that $\tilde{\nu}_T = \tilde{\nu}_T^{+1} = -\tilde{\nu}_T^{-1}$. For the considered Kernel function $\tilde{\nu}_T$ is positive definite as α is smaller than β . Therefore $\tilde{\mathfrak{A}}$ results in

$$\tilde{\mathfrak{A}} = \lim_{t \rightarrow \infty} \mathfrak{A}(t) = \begin{pmatrix} 0 & \tilde{\nu}_T \\ -\tilde{\nu}_T & 0 \end{pmatrix} = \nu_T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (15)$$

The diagonal elements become zero as the chosen Kernel function decays to zero in the limit to infinity.

As the square is $\tilde{\mathfrak{A}}^2 = -\tilde{\nu}_T^2 \mathbf{I}$ we further calculate the exponential solution Eq. 8 for an error of order ($k = 2$).

The exponential function is then:

$$\begin{aligned} \tilde{\mathfrak{B}}_{(2)} &= \exp \tilde{\mathfrak{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{A}^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \tilde{\nu}_T^{2n} \mathbf{I} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \tilde{\nu}_T^{2n+1} \mathbf{J} \quad (16) \\ &= \cos \tilde{\nu}(T) \mathbf{I} + \sin \tilde{\nu}(T) \mathbf{J} = \begin{pmatrix} \cos(\tilde{\nu}_T) & \sin(\tilde{\nu}_T) \\ -\sin(\tilde{\nu}_T) & \cos(\tilde{\nu}_T) \end{pmatrix} \end{aligned}$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This results into

$$\tilde{\omega}_{(2)} = \tilde{\mathfrak{B}}_{(2)} \cdot \omega_0 = \begin{pmatrix} \cos(\tilde{\nu}_T) & \sin(\tilde{\nu}_T) \\ -\sin(\tilde{\nu}_T) & \cos(\tilde{\nu}_T) \end{pmatrix} \omega_0. \quad (17)$$

Both, Eq. 16 and Eq. 17, were used to calculate the difference $\Delta_{(c)}$ for different values of T in Fig. 2.

* Electronic address: kolo@bccn-goettingen.de

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